Decoherence induced by scattering: a three-dimensional model

2005 J. Phys. A: Math. Gen. 38 4933

(http://iopscience.iop.org/0305-4470/38/22/016)

Institute of Physics Publishing Journal of Physics A: Mathematical and General

J. Phys. A: Math. Gen. **38** (2005) 4933–4946 doi:10.1088/0305-4470/38/22/016

Decoherence induced by scattering: a three-dimensional model

Claudio Cacciapuoti[1,2], **Raffaele Carlone**[1] **and Rodolfo Figari**[2,1]

[1] Dipartimento di Scienze Fisiche, Universitá di Napoli Federico II, Via Cintia 80126 Napoli, Italy
[2] Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Italy

E-mail: figari@na.infn.it, carlone@na.infn.it and caccia@na.infn.it

Received 30 October 2004, in final form 10 January 2005
Published 18 May 2005
Online at stacks.iop.org/JPhysA/38/4933

Abstract

The dynamical process of generation of entanglement between subsystems of an evolving quantum system is analysed in this paper. We consider one of the simplest possible continuous systems in which such a process can be rigorously studied: a quantum 'heavy' particle (the system) interacting via zero range forces with a lighter particle (the environment). In a three-dimensional model we study the asymptotic dynamics when the ratio between the masses of the light and the heavy particle is very small. The effect of decoherence on the heavy particle induced by the interaction is explicitly computed.

PACS numbers: 02.30.Tb, 03.65.Nk, 03.65.Yz

1. Introduction

The aim of this paper is to give a rigorous treatment of the asymptotic dynamics of a quantum particle undergoing a single scattering event with a much lighter particle. A detailed knowledge of such a process is the necessary preliminary step for the formulation of more realistic models for the dynamics of a quantum particle evolving in an environment made up of many light particles. In this perspective this problem was investigated by Joos and Zeh [1] first and by many others ([2–6] and references therein) successively.

Starting from a dynamical hypothesis about a single scattering event, since then referred to as Joos and Zeh formula, those authors deduced a master equation for the reduced density matrix of the heavy particle, from where they computed the characteristic times of the processes of decoherence and dissipation induced by the interaction.

Joos and Zeh noted that as a consequence of a small mass ratio two time scales characterize the evolution of the two particles: a slow one relative to the heavy particle and a much faster one relative to the light particle.

In order to specify the details of their idea let us suppose that the state of the two-particle system is initially given in a product form of the type $\varphi(R)\chi(r)$ where R and r describe respectively the spatial coordinates of the heavy particle and of the light one. The authors proposed that, in the roughest approximation, the scattering process would be described by the instantaneous transition

$$\varphi(R)\chi(r) \to \varphi(R)(S^R\chi)(r) \tag{1}$$

where S^R is the scattering operator for the light particle corresponding to the heavy one fixed at the position R. The R dependence of the scattering operator indicates that entanglement has taken place in the sense that the state of the scattered light particle keeps track of the position of the heavy one.

Details of the process of entanglement dynamically induced by a single scattering event, outlined above, were analysed in a series of recent papers ([7–9]) for different models of two-body interaction. In [8, 9] the authors gave rigorous estimates of the asymptotic dynamics, in the limit of a small mass ratio, for particles interacting respectively via a point interaction in dimension one and for a class of smooth potential in three dimensions. Their results can be considered as a rigorous formulation of the Joos and Zeh formula (1).

In this paper we will give a detailed analysis of the dynamics of a three-dimensional system made up of two quantum particles interacting via a repulsive δ-like potential. In order to define the model we need to introduce some notation and to recall few results concerning point interactions in three dimensions. As is well known a sequence of Schrödinger operators with potentials approximating a distribution supported by a single point generically tends to the free Laplacian in dimension greater than one. A different renormalization procedure has to be chosen to define a non-trivial zero range potential Hamiltonian. A general method to obtain such a kind of Hamiltonian is to analyse all the possible self-adjoint extensions of the Laplacian restricted to functions vanishing in a neighbourhood of the diffusion centre (see e.g. [10] for details). In dimension three different self-adjoint extensions are characterized by singular boundary conditions at the position of the interaction centre in the way concisely described below.

In $L^2(\mathbb{R}^3)$ there is a family of self-adjoint extensions $H_{\alpha,y}$ of the Laplacian restricted to the subspace $C_0^\infty(\mathbb{R}^3\backslash\{y\})$ of the infinitely differentiable functions with compact support not containing a single point $y \in \mathbb{R}^3$. Each operator in this family is indexed by a real parameter $-\infty < \alpha < +\infty$ and it is defined in the following way: its domain is

$$\begin{aligned}\mathcal{D}(H_{\alpha,y}) = \big\{\psi \in L^2(\mathbb{R}^3) | \psi = \phi_\lambda + qG_0(\cdot - y), \phi \in H^2_{loc}(\mathbb{R}^3),\\ \nabla\phi \in L^2(\mathbb{R}^3), \Delta\phi \in L^2(\mathbb{R}^3), q \in \mathbb{C}, \lim_{x\to y}(\psi(x) - qG_0(x-y)) = \alpha q\big\}\end{aligned} \tag{2}$$

where G_0 is the Green function for the free Laplacian for $\lambda = 0$,

$$G_\lambda(x - x') = (-\Delta + \lambda)^{-1}(x - x') = \frac{\mathrm{e}^{-\sqrt{\lambda}|x-x'|}}{4\pi|x - x'|}, \qquad \lambda \geqslant 0. \tag{3}$$

The action of $H_{\alpha,y}$ on $\mathcal{D}(H_{\alpha,y})$ is

$$(H_{\alpha,y} + \lambda)\psi = (-\Delta + \lambda)\phi_\lambda. \tag{4}$$

The corresponding resolvent operator is the rank-one perturbation of the free resolvent whose integral kernel is

$$(H_{\alpha,y} + \lambda)^{-1}(x, x') = \frac{\mathrm{e}^{-\sqrt{\lambda}|x-x'|}}{4\pi|x - x'|} + \left(\alpha + \frac{\sqrt{\lambda}}{4\pi}\right)^{-1} \frac{\mathrm{e}^{-\sqrt{\lambda}|x-y|}}{4\pi|x - y|}\frac{\mathrm{e}^{-\sqrt{\lambda}|y-x'|}}{4\pi|y - x'|}. \tag{5}$$

From the explicit form (5) of the resolvent, it is clear that the free Laplacian is obtained when $|\alpha| \to \infty$ and that the 'strength' of interaction reaches its maximum for $\alpha = 0$. A detailed analysis (see e.g. [10]) shows in fact that $-(4\pi\alpha)^{-1}$ represents the scattering length relative to $H_{\alpha,y}$.

Analysing the singularities of resolvent (5) one can easily find the spectral structure of $H_{\alpha,y}$. For any α and y the essential spectrum is purely absolutely continuous and covers the nonnegative real axis

$$\sigma_{\mathrm{ess}}(H_{\alpha,y}) = \sigma_{ac}(H_{\alpha,y}) = [0,\infty). \tag{6}$$

If $\alpha < 0$ the point spectrum of $H_{\alpha,y}$ consists of a single point $\sigma_p(H_{\alpha,y}) = -(4\pi\alpha)^2$. For $\alpha \geqslant 0$ (the only case we are going to consider in the following) the operator $H_{\alpha,y}$ has no eigenvalues, i.e. $\sigma_p(H_{\alpha,y}) = \varnothing$.

In order to simplify notation we will use H_α instead of $H_{\alpha,0}$. By inverse-Laplace transforming resolvent (5) it is possible to obtain the explicit form of the propagator $\mathrm{e}^{-\mathrm{i}tH_\alpha}$ of H_α (see [14, 15])

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}tH_\alpha}(x,y) &= \mathrm{e}^{-\mathrm{i}tH_0}(x-y) + \frac{2\mathrm{i}t}{|x||y|}\mathrm{e}^{-\mathrm{i}tH_0}(|x|+|y|)\\ &\quad - \frac{8\pi\alpha\mathrm{i}t}{|x||y|}\int_0^\infty \mathrm{e}^{-4\pi\alpha u}\,\mathrm{e}^{-\mathrm{i}tH_0}(|x|+|y|+u)\,\mathrm{d}u\end{aligned} \tag{7}$$

where $\mathrm{e}^{-\mathrm{i}tH_0}$ is the free propagator.

For every $k \in \mathbb{R}^3$ the generalized eigenfunction of $H_{\alpha,y}$ corresponding to the energy $E = |k|^2$ in the continuous spectrum is given in closed form by

$$\Phi^y_\pm(x,k) = \mathrm{e}^{\mathrm{i}kx} + \frac{\mathrm{e}^{\mathrm{i}ky}}{4\pi\alpha \pm \mathrm{i}|k|}\frac{\mathrm{e}^{\mp\mathrm{i}|k||x-y|}}{|x-y|}. \tag{8}$$

Using the generalized eigenfunctions $\Phi^y_\pm$, it is possible to define the unitary maps (see e.g. [13]) $\mathcal{F}^y_\pm : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$

$$\left[\mathcal{F}^y_\pm f\right](k) = \mathrm{s}-\lim_{R\to\infty}\frac{1}{(2\pi)^{3/2}}\int_{B_R}\overline{\Phi^y_\pm(x,k)}\,f(x)\,\mathrm{d}x \tag{9}$$

where B_R indicates the sphere of radius R in $\mathbb{R}^3$. The wave operators (see e.g. [11, 12]) for the Hamiltonian $H_{\alpha,y}$

$$\Omega^y_\pm = \mathrm{s}-\lim_{\tau\to\pm\infty}\mathrm{e}^{\mathrm{i}\tau H_{\alpha,y}}\,\mathrm{e}^{-\mathrm{i}\tau H_0} \tag{10}$$

are unitary for $\alpha > 0$ and are related to $\mathcal{F}^y_\pm$ by

$$\Omega^y_\pm = \left(\mathcal{F}^y_\pm\right)^{-1}\mathcal{F}, \qquad \left(\Omega^y_\pm\right)^{-1} = \mathcal{F}^{-1}\mathcal{F}^y_\pm \tag{11}$$

where $\mathcal{F}$ indicates the usual Fourier transform. Now we have all the ingredients to define our two-particle model. To simplify notation we fix $M = 1$ and $\hbar = 1$ and we define $\varepsilon \equiv \frac{m}{M}$. In the system of coordinates of the centre of mass $x \equiv \frac{R+\varepsilon r}{1+\varepsilon}$ and of the relative coordinate $y \equiv r - R$, the Hamiltonian for a three-dimensional system of two particles interacting via point interaction in $L^2(\mathbb{R}^3,\mathrm{d}x)\otimes L^2(\mathbb{R}^3,\mathrm{d}y)$ reads

$$H^\varepsilon = H^\nu_0 \otimes H^\mu_\alpha \tag{12}$$

where $\nu = (1+\varepsilon)$ is the total mass of the system, $\mu = \frac{\varepsilon}{1+\varepsilon}$ is the reduced mass and H^ν_0 indicates the free Hamiltonian relative to a particle of mass ν. Note that in (12) with H^μ_α we mean $\frac{1}{\mu}H_\alpha$ suggesting that a rescaling of the coupling constant α has been made (compare with the cases of two-body potentials [8, 9]).

We consider the problem

$$\mathrm{i}\frac{\partial\Psi^\varepsilon(t)}{\partial t} = H^\varepsilon\Psi^\varepsilon(t) \tag{13}$$

$$\Psi^{\varepsilon}(0; R, r) = \varphi(R)\chi(r) \tag{14}$$

in the limit of small ε. Because of the particular initial conditions (14) the positions of the two particles are uncorrelated at time zero. Nevertheless the dynamics is not factorized with respect to the coordinates R and r. The mutual interaction of the two particles, described by the static δ-like potential in the relative coordinate, will eventually produce correlations between the positions of the two particles. Our main result is expressed in the following theorem where we indicate by $\|\cdot\|$ the $L^2(\mathbb{R}^6)$-norm.

Theorem 1.1. *There exist two constants $A > 0$ and $B > 0$ such that for any initial state (14) and any fixed $\alpha > 0$ and $t > 0$, one has*

$$\|\Psi^{\varepsilon}(t) - \Psi^{a}(t)\| \leqslant A\left(\frac{\varepsilon}{t}\right)^{\frac{3}{4}} + B\varepsilon \tag{15}$$

where

$$\Psi^{a}(t) = \mathrm{e}^{-\mathrm{i}tH_0^{\varepsilon}}\Psi_0^{a} \tag{16}$$

$$H_0^{\varepsilon} = H_0 \otimes \frac{1}{\varepsilon}H_0 \tag{17}$$

$$\Psi_0^{a}(R, r) = \varphi(R)\left[\left(\Omega_{+}^{R}\right)^{-1}\chi\right](r) \tag{18}$$

and the constants A and B depend only on the initial state (see below for details) and on the constant α.

The result of theorem 1.1 expressed by (16)–(18) can be thought as an exact formulation of the Joos and Zeh conjecture (1) for the special case of point interactions in three dimensions. As stressed by many authors (see e.g. [3, 4, 8, 9],) formula (1) cannot be correct, as it stands, inasmuch as one is looking for a relation between initial and scattering states and not between *in* and *out* states. Roughly speaking (18) shows that the approximation formula holds true if in (1) the scattering matrix S^R is replaced with the wave operator $\left(\Omega_{+}^{R}\right)^{-1}$.

The specific dependence of constants A and B in (15) on the initial state will be analysed in the following section. Expressed in the language of weighted Sobolev spaces $H^{m,s}$ (see e.g. [16])

$$H^{m,s}(\mathbb{R}^d) \equiv \{u \in L^2(\mathbb{R}^d) : \|(1 + |\cdot|^2)^{\frac{s}{2}}(1 - \Delta)^{\frac{m}{2}}u\|_{L^2(\mathbb{R}^d)} < +\infty\}$$

with $L_s^2 = H^{0,s}$ and $H^m = H^{m,0}$, we will assume that initial state satisfies

Condition A. $\varphi(R) \in H^{1,1}(\mathbb{R}^3)$ and $\chi(r) \in H^{1,1}(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$.

2. Proof of the theorem 1.1

Following the same line as in [8, 9] we prove theorem 1.1 in three steps, each one consisting of the proof of a lemma.

Lemma 2.1. *If condition A is satisfied then there exists a constant $C_1 > 0$ such that, for any $t \in \mathbb{R}^{+}$, one has*

$$\left\|\Psi^{\varepsilon}(t) - \Psi_1^{\varepsilon}(t)\right\| \leqslant C_1\varepsilon \tag{19}$$

where we defined

$$\Psi_1^{\varepsilon}(t; R, r) \equiv \int_{\mathbb{R}^6} \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{e}^{-\mathrm{i}\frac{t}{1+\varepsilon}H_0}\left(\frac{R + \varepsilon r}{1 + \varepsilon} - x'\right)\varphi(x')\,\mathrm{e}^{-\mathrm{i}\frac{t(1+\varepsilon)}{\varepsilon}H_\alpha}(r - R, y')\chi(y' + x'). \tag{20}$$

Proof. Note that $\Psi^\varepsilon(t)$ is the result of the evolution generated by the Hamiltonian H^ε of the initial state $\Psi(0; x, y) = \varphi\big(x - \frac{\varepsilon y}{1+\varepsilon}\big)\chi\big(x + \frac{y}{1+\varepsilon}\big)$. Making use of the unitarity of the evolution we obtain

$$\big\|\Psi^\varepsilon(t) - \Psi_1^\varepsilon(t)\big\|^2 = \big\|\Psi^\varepsilon(0) - \Psi_1^\varepsilon(0)\big\|^2 \tag{21}$$

$$= \int_{\mathbb{R}^6} \mathrm{d}x\,\mathrm{d}y \left|\varphi\left(x - \frac{\varepsilon y}{1+\varepsilon}\right)\chi\left(x + \frac{y}{1+\varepsilon}\right) - \varphi(x)\chi(x+y)\right|^2. \tag{22}$$

We get then the following estimate:

$$\big\|\Psi^\varepsilon(0) - \Psi_1^\varepsilon(0)\big\|^2 \leqslant \varepsilon^2 \int_{\mathbb{R}^6} \mathrm{d}x\,\mathrm{d}y |y|^2 |\nabla_x(\varphi(x)\chi(x+y))|^2. \tag{23}$$

The rhs of the last inequality is finite for $\varphi \in H^{1,1}(\mathbb{R}^3)$ and $\chi \in H^{1,1}(\mathbb{R}^3)$ and the proof is completed with

$$C_1^2 \equiv \int_{\mathbb{R}^6} \mathrm{d}x\,\mathrm{d}y |y|^2 |\nabla_x(\varphi(x)\chi(x+y))|^2. \tag{24}$$

□

As we mentioned before the evolution of the system in the limit of small ε has two different time scales. In the second lemma we quantify this statement giving a rigorous estimate of how much the free evolution of the scattering state $\big[\Omega_+^{-1}\chi\big](y)$ approximates the exact evolution.

Lemma 2.2. *If condition A is satisfied then there exists a constant $C_2 > 0$ such that for any $t > 0$ one has*

$$\big\|\Psi_2^\varepsilon(t) - \Psi_1^\varepsilon(t)\big\| \leqslant C_2 \left(\frac{\varepsilon}{t}\right)^{\frac{3}{4}} \tag{25}$$

where

$$\Psi_2^\varepsilon(t; R, r) \equiv \int_{\mathbb{R}^3} \mathrm{d}x'\, \mathrm{e}^{-\mathrm{i}\frac{t}{1+\varepsilon}H_0}\left(\frac{R+\varepsilon r}{1+\varepsilon} - x'\right)\varphi(x') \tag{26}$$

$$\times \int_{\mathbb{R}^3} \mathrm{d}y'\, \mathrm{e}^{-\mathrm{i}\frac{1+\varepsilon}{\varepsilon}tH_0}(r - R - y')\big[\Omega_+^{-1}\chi(\cdot + x')\big](y'). \tag{26}$$

Proof. Following the notation of [9] we define $\chi_x(y) \equiv \chi(x+y)$. By direct computation we have

$$\big\|\Psi_2^\varepsilon(t) - \Psi_1^\varepsilon(t)\big\|^2 = \int_{\mathbb{R}^6} \mathrm{d}x\,\mathrm{d}y \Bigg| \int_{\mathbb{R}^6} \mathrm{d}x'\,\mathrm{d}y'\, \mathrm{e}^{-\mathrm{i}\frac{t}{1+\varepsilon}H_0}(x - x')\varphi(x')$$

$$\times\big[\mathrm{e}^{-\mathrm{i}\frac{1+\varepsilon}{\varepsilon}tH_0}(y - y')\big[\Omega_+^{-1}\chi_{x'}\big](y') - \mathrm{e}^{-\mathrm{i}\frac{1+\varepsilon}{\varepsilon}tH_\alpha}(y, y')\chi_{x'}(y')\big]\Bigg|^2. \tag{27}$$

Define the unitary operator $\Omega_\tau^+ = \mathrm{e}^{\mathrm{i}\tau H_\alpha}\mathrm{e}^{-\mathrm{i}\tau H_0}$ and its inverse $\big(\Omega_\tau^+\big)^{-1} = \mathrm{e}^{\mathrm{i}\tau H_0}\,\mathrm{e}^{-\mathrm{i}\tau H_\alpha}$. Using the unitarity of the free propagator $\mathrm{e}^{-\mathrm{i}tH_0}$ we obtain

$$\big\|\Psi_2^\varepsilon(t) - \Psi_1^\varepsilon(t)\big\|^2 = \int_{\mathbb{R}^3} \mathrm{d}x |\varphi(x)|^2 \int_{\mathbb{R}^3} \mathrm{d}y \left|\big[\Omega_+^{-1}\chi_x\big](y) - \left[\big(\Omega_{\frac{1+\varepsilon}{\varepsilon}t}^+\big)^{-1}\chi_x\right](y)\right|^2. \tag{28}$$

Due to the unitarity of the operators Ω_τ^+ and Ω_+ we have

$$\big\|\big(\Omega_+^{-1} - \big(\Omega_\tau^+\big)^{-1}\big)\chi\big\|_{L^2(\mathbb{R}^3)} = \big\|\big(\Omega_+ - \Omega_\tau^+\big)\Omega_+^{-1}\chi\big\|_{L^2(\mathbb{R}^3)}. \tag{29}$$

In the following we will prove that for any $\eta \in L^2_2(\mathbb{R}^3)$

$$\left\|\left(\Omega_+ - \Omega^+_\tau\right)\eta\right\|_{L^2(\mathbb{R}^3)} \leqslant \frac{C'}{\tau^{\frac{3}{4}}} \qquad \text{for} \quad \tau \to \infty. \tag{30}$$

In fact from (11) we have

$$[\Omega_+\eta](x) = \left[\mathcal{F}_+^{-1}\mathcal{F}\eta\right](x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^6} \mathrm{d}k\,\mathrm{d}y\,\Phi_+(x,k)\,\mathrm{e}^{-\mathrm{i}ky}\eta(y). \tag{31}$$

whereas from its definition

$$\left[\Omega^+_\tau\eta\right](y) = \int_{\mathbb{R}^6} \mathrm{d}z\,\mathrm{d}y'\,\mathrm{e}^{\mathrm{i}\tau H_\alpha}(y,z)\,\mathrm{e}^{-\mathrm{i}\tau H_0}(z-y')\eta(y'). \tag{32}$$

By explicit computation we have

$$\left(\Omega_+ - \Omega^+_\tau\right)\eta = W_0\eta + W_\alpha\eta \tag{33}$$

with

$$[W_0\eta](|x|) = \frac{2\mathrm{i}}{(2\pi)^2}\frac{1}{|x|}\int_0^\infty \frac{1-\mathrm{e}^{-\mathrm{i}\frac{|x|^2-|y|^2}{4\tau}}}{|x|^2-|y|^2}g(|y|)\,\mathrm{d}|y| \tag{34}$$

and

$$\begin{aligned}[W_\alpha\eta](|x|) = {}& \frac{8\pi\mathrm{i}\alpha}{(2\pi)^2}\frac{1}{|x|}\int_0^\infty \mathrm{d}|y|\frac{g(|y|)}{|y|}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}s|x|}\sin s|y| \\ & \times\left(\frac{1}{4\pi\alpha+\mathrm{i}s} - \mathrm{e}^{-\mathrm{i}\frac{|x|^2-|y|^2}{4\tau}}\sqrt{-\mathrm{i}\pi\tau}\,\mathrm{e}^{z^2}\mathrm{erfc}(z)\right)\end{aligned} \tag{35}$$

with

$$z = \sqrt{-\mathrm{i}\tau}\left(4\pi\alpha + \mathrm{i}s + \mathrm{i}\frac{|x|}{2\tau}\right) \qquad \text{and} \qquad g(|x|) = |x|^2\int \eta(|x|, x_\theta, x_\varphi)\,\mathrm{d}\Omega_x.$$

We start with an estimate for W_0. From (34) we have

$$\|W_0\eta\|^2_{L^2(\mathbb{R}^3)} \leqslant \frac{16}{(2\pi)^3}\int_0^\infty |g(|y|)|^2 K_\tau(|y|)\,\mathrm{d}|y| \tag{36}$$

where

$$K_\tau(|y|) = \frac{1}{16\tau^{\frac{3}{2}}}\int_0^\infty \frac{1-\cos\left(\xi - \frac{|y|^2}{4\tau}\right)}{\left(\xi - \frac{|y|^2}{4\tau}\right)^2}\frac{1}{\sqrt{\xi}}\,\mathrm{d}\xi \leqslant \frac{1}{16\tau^{\frac{3}{2}}}\left[2 + \sqrt{\frac{|y|^2}{4\tau}+1}\right]. \tag{37}$$

We obtain then

$$\|W_0\eta\|^2_{L^2(\mathbb{R}^3)} \leqslant D_0\left[\frac{1}{\tau^{\frac{3}{2}}}\int_0^\infty |g(|y|)|^2\,\mathrm{d}|y| + \frac{1}{\tau^2}\int_0^\infty |g(|y|)|^2|y|\mathrm{d}|y|\right]. \tag{38}$$

The estimate for the term $W_\alpha\eta$ in (35) will be given in few steps. We write $W_\alpha\eta$ as the sum of four terms

$$\begin{aligned}[W_\alpha\eta](|x|) = {}& \frac{8\pi\mathrm{i}\alpha}{(2\pi)^2}\frac{1}{|x|}\int_0^\infty \mathrm{d}|y|\frac{g(|y|)}{|y|}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}s|x|}\sin s|y|\,\mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}} \\ & \times\left[\frac{\mathrm{e}^{-\mathrm{i}\frac{|y|^2}{4\tau}}-1}{4\pi\alpha+\mathrm{i}s} + \frac{1-\mathrm{e}^{-\mathrm{i}\frac{|x|^2}{4\tau}}}{4\pi\alpha+\mathrm{i}s} + \mathrm{e}^{-\mathrm{i}\frac{|x|^2}{4\tau}}\left(\frac{1}{4\pi\alpha+\mathrm{i}s} - \frac{1}{4\pi\alpha+\mathrm{i}s+\mathrm{i}\frac{|x|}{2\tau}}\right)\right. \\ & \left. - \mathrm{e}^{-\mathrm{i}\frac{|x|^2}{4\tau}}\left(\sqrt{-\mathrm{i}\pi\tau}\,\mathrm{e}^{z^2}\mathrm{erfc}(z) - \frac{1}{4\pi\alpha+\mathrm{i}s+\mathrm{i}\frac{|x|}{2\tau}}\right)\right].\end{aligned} \tag{39}$$

We have then

$$\|W_\alpha \eta\|^2_{L^2(\mathbb{R}^3)} \leqslant W_1 + W_2 + W_3 + W_4 \tag{40}$$

with

$$W_1 = D\int_0^\infty \mathrm{d}|x| \left| \int_0^\infty \mathrm{d}s \frac{\mathrm{e}^{-\mathrm{i}s|x|}}{4\pi\alpha + \mathrm{i}s} \mathcal{S}\left(\frac{g(|y|)}{|y|}\left(1 - \mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}\right), |y|\right)(s) \right|^2 \tag{41}$$

$$W_2 = 2D\int_0^\infty \mathrm{d}|x| \left(1 - \cos\frac{|x|^2}{4\tau}\right) \left| \int_0^\infty \mathrm{d}s \frac{\mathrm{e}^{-\mathrm{i}s|x|}}{4\pi\alpha + \mathrm{i}s} \mathcal{S}\left(\frac{g(|y|)}{|y|}\mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}, |y|\right)(s) \right|^2 \tag{42}$$

$$W_3 = \frac{D}{4\tau^2}\int_0^\infty \mathrm{d}|x||x|^2 \left| \int_0^\infty \mathrm{d}s \frac{\mathrm{e}^{-\mathrm{i}s|x|}}{(4\pi\alpha + \mathrm{i}s)\left(4\pi\alpha + \mathrm{i}s + \mathrm{i}\frac{|x|}{2\tau}\right)} \mathcal{S}\left(\frac{g(|y|)}{|y|}\,\mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}, |y|\right)(s) \right|^2 \tag{43}$$

$$\begin{aligned} W_4 = D\int_0^\infty \mathrm{d}|x| \Bigg| \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}s|x|} &\left(\sqrt{-\mathrm{i}\pi\tau}\,\mathrm{e}^{z^2}\operatorname{erfc}(z) - \frac{1}{4\pi\alpha + \mathrm{i}s + \mathrm{i}\frac{|x|}{2\tau}} \right) \\ &\times \mathcal{S}\left(\frac{g(|y|)}{|y|}\mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}, |y|\right)(s) \Bigg|^2 \end{aligned} \tag{44}$$

where

$$D = \frac{16\alpha^2}{\pi} \qquad \text{and} \qquad \mathcal{S}(f(|y|), |y|)(s) = \int_0^\infty \sin s|y| f(|y|)\,\mathrm{d}|y|$$

is the Fourier sin transform of $f(|y|)$. Let us define

$$h(s) = \begin{cases} \dfrac{1}{4\pi\alpha + \mathrm{i}s} \mathcal{S}\left(\dfrac{g(|y|)}{|y|}\left(1 - \mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}\right), |y|\right)(s) & s \geqslant 0 \\ 0 & s < 0 \end{cases} \tag{45}$$

so that

$$W_1 = 2\pi D\|\hat{h}\|^2_{L^2((0,\infty))} \leqslant 2\pi D\|\hat{h}\|^2_{L^2(\mathbb{R})} = 2\pi D\,\|h\|^2_{L^2(\mathbb{R})},$$

where $\hat{h}$ is the usual one-dimensional Fourier transform of $h(s)$. A straightforward computation gives

$$W_1 \leqslant \frac{D_1}{\tau^2}\int_0^\infty |g(|y|)|^2\,|y|^2\,\mathrm{d}|y|. \tag{46}$$

It is easily seen from the definition of W_2 that

$$W_2 = 2D\int_0^\infty \mathrm{d}|x| \frac{1 - \cos\frac{|x|^2}{4\tau}}{(1 + |x|^2)^2} \left| \int_0^\infty \mathrm{d}s \frac{\left(1 - \frac{\mathrm{d}^2}{\mathrm{d}s^2}\right)\mathrm{e}^{-\mathrm{i}s|x|}}{4\pi\alpha + \mathrm{i}s} \mathcal{S}\left(\frac{g(|y|)}{|y|}\mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}, |y|\right)(s) \right|^2. \tag{47}$$

An integration by parts in the variable s and an estimate of the integral in the variable $|x|$ for large τ give

$$W_2 \leqslant \frac{D}{\tau^{\frac{3}{2}}}\left\{ \frac{1}{(4\pi\alpha)^2}\int_0^\infty \mathrm{d}|y||g(|y|)|^2 + \int_0^\infty \mathrm{d}s \left| \left(1 - \frac{\mathrm{d}^2}{\mathrm{d}s^2}\right) \frac{\mathcal{S}\left(\frac{g(|y|)}{|y|}\,\mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}, |y|\right)(s)}{(4\pi\alpha + \mathrm{i}s)} \right|^2 \right\}. \tag{48}$$

We rewrite W_3 in the following way:

$$W_3 = \frac{D}{4\tau^2}\int_0^\infty \mathrm{d}|x| \left|\frac{|x|}{1+|x|^2}\right|^2 \left|\int_0^\infty \mathrm{d}s \frac{(1-\frac{\mathrm{d}^2}{\mathrm{d}s^2})\,\mathrm{e}^{-\mathrm{i}s|x|}}{(4\pi\alpha+\mathrm{i}s)\big(4\pi\alpha+\mathrm{i}s+\mathrm{i}\frac{|x|}{2\tau}\big)} \right.$$
$$\left. \times\, \mathcal{S}\left(\frac{g(|y|)}{|y|}\mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}, |y|\right)(s)\right|^2 \tag{49}$$

and we use the inequality

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}s^m}\frac{1}{4\pi\alpha+\mathrm{i}s+\mathrm{i}\frac{|x|}{2\tau}}\right|^2 \leqslant \left|\frac{\mathrm{d}^m}{\mathrm{d}s^m}\frac{1}{4\pi\alpha+\mathrm{i}s}\right|^2 \qquad \forall\tau\geqslant 0,\quad \forall m\in\mathbb{N}_0\quad\text{and}\quad\forall x\in\mathbb{R}^3,$$

to obtain

$$W_3 \leqslant \frac{D}{\tau^2}\left\{\frac{1}{(4\pi\alpha)^4}\int_0^\infty \mathrm{d}|y||g(|y|)|^2 + \int_0^\infty \mathrm{d}s\left|\left(1-\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right)\frac{\mathcal{S}\left(\frac{g(|y|)}{|y|}\,\mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}, |y|\right)(s)}{(4\pi\alpha+\mathrm{i}s)^2}\right|^2\right\}. \tag{50}$$

In the W_4 term for $\tau\to\infty$ we have $|z|\to\infty$ and we can use the asymptotic expansion

$$\mathrm{e}^{z^2}\mathrm{erfc}(z) - \frac{1}{\sqrt{\pi}z} = -\frac{1}{2\sqrt{\pi}z^3} + o\left(\frac{1}{z^5}\right).$$

From the inequality

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}z^m}\left(\mathrm{e}^{z^2}\mathrm{erfc}(z)-\frac{1}{\sqrt{\pi}z}\right)\right| \leqslant \left|\frac{1}{\sqrt{\pi}z^{3+m}}\right| \qquad \text{for}\quad |z|\to\infty \tag{51}$$

$\forall m\in\mathbb{N}_0$, we obtain

$$W_4 \leqslant \frac{D}{\tau^2}\int_0^\infty \mathrm{d}|x|\frac{1}{1+|x|^2}\int_0^\infty \mathrm{d}s\left|\left(1-\frac{\mathrm{d}}{\mathrm{d}s}\right)\frac{1}{\big(4\pi\alpha+\mathrm{i}s+\mathrm{i}\frac{|x|}{2\tau}\big)^3}\mathcal{S}\left(\frac{g(|y|)}{|y|}\,\mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}, |y|\right)(s)\right|^2 \tag{52}$$

With the same estimate used in (49) it is easily seen that

$$W_4 \leqslant \frac{\pi D}{2\tau^2}\int_0^\infty \mathrm{d}s\left|\left(1-\frac{\mathrm{d}}{\mathrm{d}s}\right)\frac{1}{(4\pi\alpha+\mathrm{i}s)^3}\mathcal{S}\left(\frac{g(|y|)}{|y|}\mathrm{e}^{\mathrm{i}\frac{|y|^2}{4\tau}}, |y|\right)(s)\right|^2. \tag{53}$$

Note that if $\eta\in L_2^2(\mathbb{R}^3)$ all the integrals in (38), (46), (48), (50), (53) are finite and we get estimate (30). From (28) and (29) in order to conclude the proof of lemma 2.2 we need to show that if the initial state satisfies condition A then $\eta = \Omega_+^{-1}\chi_x\in L_2^2(\mathbb{R}^3)$ for every $x\in\mathbb{R}^3$ and

$$\left\|\Omega_+^{-1}\chi_x\right\|_{L_2^2(\mathbb{R}^3)} \leqslant C'(1+|x|^2)^{\frac{1}{2}}. \tag{54}$$

We omit the details of this last result, which follows easily from an integration by parts in the explicit definition of the L_2^2 norm of $\Omega_+^{-1}\chi_x$. □

To conclude the proof of theorem 1.1 we will show that the evolution of the initial state $\varphi(x)\big[\Omega_+^{-1}\chi_x\big](y)$ according to the dynamics generated by the Hamiltonian $\frac{1}{1+\varepsilon}H_0\otimes\frac{1+\varepsilon}{\varepsilon}H_0$ approximate at the order ε the dynamics of the initial state $\varphi(R)[(\Omega_+^R)^{-1}\chi](r)$ generated by the Hamiltonian H_0^ε.

Using the identity

$$\mathrm{e}^{-\mathrm{i}\frac{t}{1+\varepsilon}H_0}\left(\frac{\varepsilon(r-r')+(R-R')}{1+\varepsilon}\right)\mathrm{e}^{-\mathrm{i}\frac{1+\varepsilon}{\varepsilon}tH_0}(r-r'-(R-R'))$$
$$=\mathrm{e}^{-\mathrm{i}tH_0}(R-R')\,\mathrm{e}^{-\mathrm{i}\frac{t}{\varepsilon}H_0}(r-r') \tag{55}$$

we obtain

$$\Psi_2^\varepsilon(t;r,R)=\int_{\mathbb{R}^6}\mathrm{d}r'\,\mathrm{d}R'\,\mathrm{e}^{-\mathrm{i}tH_0}(R-R')\,\mathrm{e}^{-\mathrm{i}\frac{t}{\varepsilon}H_0}(r-r')$$
$$\times\varphi\left(\frac{\varepsilon r'+R'}{1+\varepsilon}\right)\left[\Omega_+^{-1}\chi\left(\frac{\varepsilon r'+R'}{1+\varepsilon}+\cdot\right)\right](r'-R'). \tag{56}$$

We prove the last lemma

Lemma 2.3. *There exists a constant $C_3>0$ such that for any $t\in\mathbb{R}$ one has*

$$\left\|\Psi_2^\varepsilon(t)-\Psi^a(t)\right\|\leqslant C_3\varepsilon \tag{57}$$

Proof. Given the unitarity of the free propagator

$$\left\|\Psi_2^\varepsilon(t)-\Psi^a(t)\right\|^2=\int_{\mathbb{R}^6}\mathrm{d}R\,\mathrm{d}r\left|\varphi\left(\frac{\varepsilon r+R}{1+\varepsilon}\right)\left[\Omega_+^{-1}\chi\left(\frac{\varepsilon r+R}{1+\varepsilon}+\cdot\right)\right](r-R)\right.$$
$$\left.-\varphi(R)\left[\Omega_+^{-1}\chi(R+\cdot)\right](r-R)\right|^2 \tag{58}$$

where we used the relation $\left[\left(\Omega_+^R\right)^{-1}\chi\right](r)=\left[\Omega_+^{-1}\chi_R\right](r-R)$. In the system of coordinates of the centre of mass this reads

$$\left\|\Psi_2^\varepsilon(t)-\Psi^a(t)\right\|^2=\int_{\mathbb{R}^6}\mathrm{d}x\,\mathrm{d}y\left|\varphi(x)\left[\Omega_+^{-1}\chi(x+\cdot)\right](y)\right.$$
$$\left.-\varphi\left(x-\frac{\varepsilon}{1+\varepsilon}y\right)\left[\Omega_+^{-1}\chi\left(x-\frac{\varepsilon}{1+\varepsilon}y+\cdot\right)\right](y)\right|^2. \tag{59}$$

In the limit of small ε we can write

$$\left\|\Psi_2^\varepsilon(t)-\Psi^a(t)\right\|^2\leqslant\varepsilon^2\int_{\mathbb{R}^6}\mathrm{d}x\,\mathrm{d}y|y|^2\left|\nabla_x\left[\varphi(x)\Omega_+^{-1}\chi_x(y)\right]\right|^2$$
$$\leqslant\varepsilon^2\left((\diamond\mathbf{1})+(\diamond\mathbf{2})\right). \tag{60}$$

Let us prove that terms $(\diamond\mathbf{1})$, $(\diamond\mathbf{2})$ in (60) are finite. Using definition (11) of Ω_+^{-1} and the explicit form of the generalized functions $\Phi_+(x,k)$ we obtain

$$(\diamond\mathbf{1})=\int_{\mathbb{R}^6}\mathrm{d}x\,\mathrm{d}y|\nabla_x\varphi(x)|^2|y|^2\left|\Omega_+^{-1}\chi_x(y)\right|^2$$
$$=\left(\frac{1}{2\pi}\right)^3\int_{\mathbb{R}^6}\mathrm{d}x\,\mathrm{d}k|\nabla_x\varphi(x)|^2\left|\nabla_k\int_{\mathbb{R}^3}\mathrm{d}z\left(\mathrm{e}^{-\mathrm{i}kz}+\frac{1}{4\pi\alpha-\mathrm{i}|k|}\frac{\mathrm{e}^{\mathrm{i}|k||z|}}{|z|}\right)\chi_x(z)\right|^2$$
$$\leqslant(\diamond\mathbf{3})+(\diamond\mathbf{4}). \tag{61}$$

The estimate of term ($\diamond\mathbf{3}$) follows easily

$$(\diamond\mathbf{3}) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^3} \mathrm{d}x |\nabla_x \varphi(x)|^2 \int_{\mathbb{R}^3} \mathrm{d}z ||z|\chi_x(z)|^2$$
$$\leqslant \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^3} \mathrm{d}x |\nabla_x \varphi(x)|^2 \int_{\mathbb{R}^3} \mathrm{d}z ||z|\chi_x(z)|$$
$$+ \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^3} \mathrm{d}x |\nabla_x \varphi(x)|^2 |x|^2 \, \|\chi(z)\|^2_{L^2(\mathbb{R}^3)} \tag{62}$$

For the term ($\diamond\mathbf{4}$) we have

$$(\diamond\mathbf{4}) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^6} \mathrm{d}x\, \mathrm{d}k |\nabla_x \varphi(x)|^2 \left| \frac{\mathrm{d}}{\mathrm{d}|k|} \int_{\mathbb{R}^3} \mathrm{d}z \frac{1}{4\pi\alpha - \mathrm{i}|k|} \frac{\mathrm{e}^{\mathrm{i}|k||z|}}{|z|} \chi_x(z) \right|^2 \leqslant (\diamond\mathbf{5}) + (\diamond\mathbf{6}) \tag{63}$$

with

$$(\diamond\mathbf{5}) = \int_{\mathbb{R}^6} \mathrm{d}x\, \mathrm{d}k \frac{|\nabla_x \varphi(x)|^2}{((4\pi\alpha)^2 + |k|^2)^2} \left| \int_{\mathbb{R}^3} \mathrm{d}z \frac{\mathrm{e}^{\mathrm{i}|k||z|}}{|z|} \chi_x(z) \right|^2 . \tag{64}$$

In (64) the only problem is represented by the integral in the variable z. Making explicit the x dependence of $\chi_x(z)$ we have

$$\left| \int_{\mathbb{R}^3} \mathrm{d}z \frac{\mathrm{e}^{\mathrm{i}|k||z|}}{|z|} \chi(z+x) \right|^2 = \left| \int_{\mathbb{R}^3} \mathrm{d}\xi \frac{\mathrm{e}^{\mathrm{i}|k||\xi - x|}}{|\xi - x|(1+|\xi|^2)^{\frac{1}{2}}} \chi(\xi)(1+|\xi|^2)^{\frac{1}{2}} \right|^2$$
$$\leqslant \left(\int_{\mathbb{R}^3} \mathrm{d}\xi |\chi(\xi)|^2 (1+|\xi|^2) \right) \left(\int_{\mathbb{R}^3} \mathrm{d}\xi \frac{1}{|\xi - x|^2(1+|\xi|^2)} \right)$$
$$= \|(1+|\cdot|^2)^{\frac{1}{2}} \chi\|^2_{L^2(\mathbb{R}^3)} \left\| \frac{1}{|\cdot|(1+|\cdot + x|^2)^{\frac{1}{2}}} \right\|^2_{L^2(\mathbb{R}^3)} \tag{65}$$

where we used Holder's inequality. An explicit computation shows that

$$\left\| \frac{1}{|\cdot|(1+|\cdot + x|^2)^{\frac{1}{2}}} \right\|^2_{L^2(\mathbb{R}^3)} \leqslant \frac{\pi^2}{|x|}. \tag{66}$$

We finally estimate the term

$$(\diamond\mathbf{6}) = \int_{\mathbb{R}^6} \mathrm{d}x\, \mathrm{d}k \frac{|\nabla_x \varphi(x)|^2}{(4\pi\alpha)^2 + |k|^2} \left| \int_{\mathbb{R}^3} \mathrm{d}z \chi_x(z)\, \mathrm{e}^{-\mathrm{i}|k||z|} \right|^2 . \tag{67}$$

To ensure convergence we need that the integral in the variable z goes to zero at infinity faster than $1/|k|^{1/2}$. In fact integrating by parts

$$\left| \int_{\mathbb{R}^3} \mathrm{d}z \chi(z+x)\, \mathrm{e}^{\mathrm{i}|k||z|} \right|^2 = \left| \int_{\mathbb{R}^3} \mathrm{d}\xi \chi(\xi)\, \mathrm{e}^{\mathrm{i}|k||\xi - x|} \right|^2 = \left| \int_{\mathbb{R}^3} \mathrm{d}\xi \chi(\xi) \frac{-\mathrm{i}}{|k|} \frac{\xi - x}{|\xi - x|} \nabla_\xi \mathrm{e}^{\mathrm{i}|k||\xi - x|} \right|^2$$
$$\leqslant \frac{1}{|k|^2} \int_{\mathbb{R}^3} \mathrm{d}\xi \left| \nabla_\xi \chi(\xi) \right|^2 + \frac{4}{|k|^2} \left| \int_{\mathbb{R}^3} \mathrm{d}\xi \frac{\mathrm{e}^{\mathrm{i}|k||\xi - x|}}{|\xi - x|} \chi(\xi) \right|^2 . \tag{68}$$

We are left to show that also term ($\diamond\mathbf{2}$)

$$(\diamond\mathbf{2}) = \int_{\mathbb{R}^6} \mathrm{d}x\, \mathrm{d}y |\varphi(x)|^2 |y|^2 \left| \nabla_x \Omega_+^{-1} \chi_x(y) \right|^2 \tag{69}$$

is finite. Note that

$$\left|\nabla_x \Omega_+^{-1} \chi_x(y)\right|^2 = \sum_{i=1}^{3} \left|\partial_{x_i}\left(\Omega_+^{-1} \chi_x\right)(y)\right|^2 = \sum_{i=1}^{3} \left|\Omega_+^{-1} f_{i,x}(y)\right|^2 \tag{70}$$

with $f_{i,x}(z) = \partial_{x_i} \chi_x(z) = \partial_{x_i} \chi(z+x) = f_i(z+x)$. It follows that the estimate for ($\diamond$**2**) can be obtained with the same procedure used for ($\diamond$**1**), the only difference being that we must replace $\chi(x+z)$ with $\nabla\chi(x+z)$. We conclude that all the integrals are finite if condition A is satisfied. □

3. Decoherence induced by scattering

As was done in [8, 9] we want to apply the results obtained in this paper to the analysis of the decoherence effects induced by a single scattering event. The often called 'naive' interpretation of decoherence in quantum systems will be the main idea behind the considerations which follow. Roughly speaking that interpretation insists on the almost obvious statement that entanglement causes a diffusion of quantum correlations out of every subsystem in interaction with a large environment.

The mechanism is essentially described as follows: suppose we have a subsystem of a large system which is initially in a pure state. Entanglement induced by the interaction of the subsystem with its environment forces the quantum correlations between local observables to migrate into the whole system. The trace over the exterior degrees of freedom partially cancels correlations making the reduced density matrix, describing the evolution of the subsystem, a statistical mixture.

In the following we will find an estimate for the effect of decoherence resulting from a single scattering event at the level of approximation of the dynamics given by the Joos and Zeh formula. As was done in the one-dimensional case [8], the estimate allows us to compute how much quantum interference observed in the evolution of the state of the heavy particle, initially in a superposition state, is decreased by the presence of the light particle. We will interpret the decreasing of interference as a sign of a more classical behaviour of the heavy particle.

The reduced density matrix for the heavy particle in the spatial coordinates representation is the positive, trace class operator $\rho^\varepsilon(t)$ in $L^2(\mathbb{R}^3)$ with $\mathrm{Tr}\rho^\varepsilon(t) = 1$ with integral kernel

$$\rho^\varepsilon(t; R, R') = \int_{\mathbb{R}^3} \mathrm{d}r\, \Psi^\varepsilon(t; R, r)\overline{\Psi^\varepsilon}(t; R', r) \tag{71}$$

where $\Psi^\varepsilon(t; R, r)$ is the solution of problem (13) and (14).

In the small mass ratio limit, using the results contained in theorem 1.1, one easily obtains the following approximation for the density matrix (71):

$$\rho^a(t) = \mathrm{e}^{-\mathrm{i}tH_0}\rho_0^a\, \mathrm{e}^{\mathrm{i}tH_0} \tag{72}$$

where

$$\rho_0^a(R, R') = \varphi(R)\overline{\varphi}(R')\mathcal{I}(R, R') \tag{73}$$

$$\mathcal{I}(R, R') = \left(\left(\Omega_+^R\right)^{-1}\chi, \left(\Omega_+^{R'}\right)^{-1}\chi\right). \tag{74}$$

It is easily seen that the following proposition holds

Proposition 3.1. *Under the same assumptions of theorem 1.1 one has*

$$\mathrm{Tr}\left|\rho^\varepsilon(t) - \rho^a(t)\right|^{\frac{1}{2}} \leqslant A\left(\frac{\varepsilon}{t}\right)^{\frac{3}{4}} + B\varepsilon \tag{75}$$

Without interaction the dynamics of the heavy particle is described by the free evolution of the density matrix $\rho_0(R, R') = \varphi(R)\overline{\varphi}(R')$. Being $\rho_0(R, R')$ a projector operator one has

$$\mathrm{Tr}(\rho(t))^2 = \mathrm{Tr}(\rho_0)^2 = 1. \tag{76}$$

The amount of entanglement due to the interaction at the order of approximation of the Joos and Zeh formula is expressed by the term $\mathcal{I}(R, R')$ in the initial density matrix. Given the unitarity of the operators $\left(\Omega_+^R\right)^{-1}$ it is obvious that for $R \neq R'$ one has $|\mathcal{I}(R, R')| < 1$. This implies that

$$\mathrm{Tr}(\rho^a(t))^2 = \mathrm{Tr}\left(\rho_0^a\right)^2 < 1 \tag{77}$$

which in turns means that the reduced density matrix (72) describes a mixed state.

In addition to these immediate consequences of the unitarity of $\left(\Omega_+^R\right)^{-1}$ it is in principle possible in our specific model to compute explicitly $\mathcal{I}(R, R')$.

Given the unitarity of the Fourier transform and the definition of $\left(\Omega_+^R\right)^{-1}$ we can write

$$\mathcal{I}(R, R') = \left(\mathcal{F}_+^R\chi, \mathcal{F}_+^{R'}\chi\right). \tag{78}$$

We introduce the notation

$$\mathcal{F}_+^R = \mathcal{F} + K_R \tag{79}$$

where $\mathcal{F}$ is the usual Fourier transform and K_R is the operator

$$[K_R\chi](|k|) = \int_{\mathbb{R}^3} \frac{\mathrm{d}r}{(2\pi)^{\frac{3}{2}}} \frac{\mathrm{e}^{-\mathrm{i}kR}}{4\pi\alpha - \mathrm{i}|k|} \frac{\mathrm{e}^{\mathrm{i}|k||r-R|}}{|r-R|}\chi(r) \tag{80}$$

with this notation

$$\mathcal{I}(R, R') = (\chi, \chi) + (K_R\chi, \mathcal{F}\chi) + (\mathcal{F}\chi, K_{R'}\chi) + (K_R\chi, K_{R'}\chi). \tag{81}$$

Note that because the unitarity of $(\Omega_+^R)^{-1}, \mathcal{I}(R, R) = (\chi, \chi)$ and (81) implies

$$(K_R\chi, \mathcal{F}\chi) = -(\mathcal{F}\chi, K_R\chi) - (K_R\chi, K_R\chi). \tag{82}$$

To get an estimate for the amount of decoherence we consider a normalized state $(\chi, \chi) = 1$ and compute the quantity $1 - \mathcal{I}(R, R')$. From (81) and (82) we obtain

$$1 - \mathcal{I}(R, R') = (\mathcal{F}\chi, (K_R - K_{R'})\chi) + (K_R\chi, (K_R - K_{R'})\chi). \tag{83}$$

We will analyse (83) in the particular relevant case in which the initial state of the light particle is given by a symmetric wave packet centred at the origin; in particular let us choose

$$\chi(r) = \frac{\mathrm{e}^{-\frac{|r|^2}{2\sigma^2}}}{(\pi\sigma^2)^{\frac{3}{4}}}. \tag{84}$$

We will address our efforts to the special case in which $R' = -R$ and we will evaluate $\mathcal{I}(R, -R)$. It is easy to see that for every state such that $\chi(r) = \chi(-r)$

$$(\mathcal{F}\chi, (K_R - K_{-R})\chi) = 0. \tag{85}$$

Under the same assumption on $\chi(r)$ the second term on the rhs of (83) can be written as

$$\begin{aligned}(K_R\chi, (K_R - K_{-R})\chi) = &\int_{\mathbb{R}^3} \frac{\mathrm{d}k}{(2\pi)^3} \frac{1 - \mathrm{e}^{2\mathrm{i}kR}}{(4\pi\alpha)^2 + |k|^2} \\ &\times \int_{\mathbb{R}^3} \mathrm{d}r \frac{\mathrm{e}^{-\mathrm{i}|k||r|}}{|r|}\overline{\chi}(r + R) \int_{\mathbb{R}^3} \mathrm{d}r' \frac{\mathrm{e}^{\mathrm{i}|k||r'|}}{|r'|}\chi(r' + R).\end{aligned} \tag{86}$$

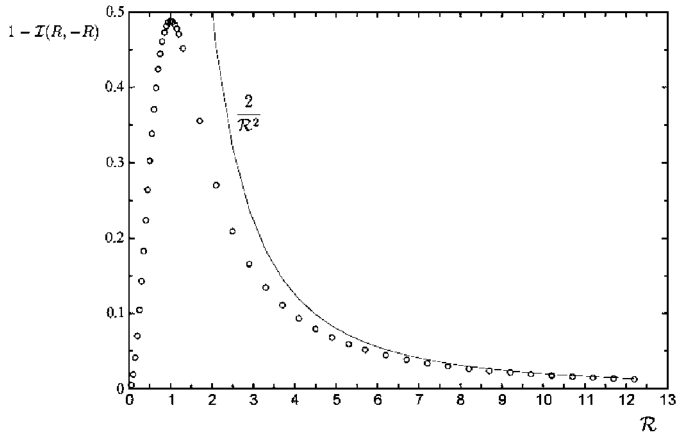

Figure 1. The dotted plot shows the numerical solution of equation (89), the continuous line represents the theoretical behaviour of large $\mathcal{R}$.

The two integrals in r and r' on the rhs of the last expression are one the complex conjugate of the other. Using the specific form (84) of $\chi(r)$ we obtain

$$\left|\int \mathrm{d}r \frac{\mathrm{e}^{-\mathrm{i}|k||r|}}{|r|}\overline{\chi}(r+R)\right|^2 = 2\pi^{\frac{3}{2}}\frac{\sigma^3}{|R|^2}\,\mathrm{e}^{-|k|^2\sigma^2}\left|\mathrm{e}^{\mathrm{i}|k||R|}\mathrm{erf}(z)+\mathrm{e}^{-\mathrm{i}|k||R|}\,\overline{\mathrm{erf}(z)}-2\mathrm{i}\sin|k||R|\right|^2 \tag{87}$$

where

$$z=\frac{|R|+\mathrm{i}|k|\sigma^2}{\sqrt{2}\sigma}.$$

Inserting this in (86) and integrating on the angular part of k we have

$$\begin{aligned}1-\mathcal{I}(R,-R) &= \frac{\sigma^3}{|R|^2\sqrt{\pi}}\int_0^\infty \mathrm{d}|k|\frac{|k|^2}{(4\pi\alpha)^2+|k|^2}\left(1-\frac{\sin(2|k||R|)}{2|k||R|}\right)\mathrm{e}^{-|k|^2\sigma^2}\\ &\quad\times\left|\mathrm{e}^{\mathrm{i}|k||R|}\mathrm{erf}(z)+\mathrm{e}^{-\mathrm{i}|k||R|}\overline{\mathrm{erf}(z)}-2\mathrm{i}\sin|k||R|\right|^2.\end{aligned} \tag{88}$$

Expression (88) clearly shows that for every R one has $1-\mathcal{I}(R,-R)\geqslant 0$, moreover it is easy to see that, for fixed R, $1-\mathcal{I}(R,-R)$ is a decreasing function of α. For this reason we focus our attention on the evaluation of (88) when $\alpha=0$.

We define the dimensionless variables $\xi\equiv|k||R|$ and $\mathcal{R}\equiv\frac{|R|}{\sigma}$. With this notation one has

$$\begin{aligned}1-\mathcal{I}(R,-R) &= \frac{1}{|\mathcal{R}|^3\sqrt{\pi}}\int_0^\infty \mathrm{d}\xi\left(1-\frac{\sin(2\xi)}{2\xi}\right)\mathrm{e}^{-\frac{\xi^2}{\mathcal{R}^2}}\\ &\quad\times\left|\mathrm{e}^{\mathrm{i}\xi}\mathrm{erf}\left(\frac{\mathcal{R}}{\sqrt{2}}+\frac{\mathrm{i}}{\sqrt{2}}\frac{\xi}{\mathcal{R}}\right)+\mathrm{e}^{-\mathrm{i}\xi}\mathrm{erf}\left(\frac{\mathcal{R}}{\sqrt{2}}-\frac{\mathrm{i}}{\sqrt{2}}\frac{\xi}{\mathcal{R}}\right)-2\mathrm{i}\sin\xi\right|^2.\end{aligned} \tag{89}$$

Analysing the asymptotics of the positive integral in (89) it is easy to check that $1-\mathcal{I}(R,-R)$ tends to zero as $1/\mathcal{R}^2$ when $\mathcal{R}$ grows to infinity, and as $\mathcal{R}$ when $\mathcal{R}$ tends to zero.

It is more interesting to investigate the range of values of $\mathcal{R}$ for which quantum interference is expected. The integral in (89) is not computable in closed form; its numerically computed behaviour as a function of the parameter $\mathcal{R}$ is given in figure 1.

Together with the initial state (84) for the light particle, let us consider an initial state of the heavy particle which is a coherent superposition of two wave packets concentrated in regions symmetrically placed around the origin, at a distance $|R|$ each one with average momentum $\pm p_0$ heading towards the origin. At a time approximatively given by the classical flight time $\frac{|R|}{|p_0|}$ one expects quantum interference to take place for distances of the order of the dispersion of the two-wave packets. Formula (73) for the approximate initial density matrix suggests that if σ is of the same order of the distance of the wave packets a maximum decoherence effect will take place.

As was mentioned in the introduction, Joos and Zeh, in their seminal paper on the subject [1], proceeded from the single scattering event towards the analysis of the decoherence effects induced on the heavy particle by the interaction with a gas of light particles.

In the case of a large number of non-interacting light particles one expects to be able to prove a generalization of theorem 1.1 in the direction suggested by Joos and Zeh. In turn this would imply a decoherence effect which is exponentially increasing with the number of the particles of the environment.

Although conceivably true on a heuristic basis, the above-mentioned result is not easy to prove, taking into account the complete Schrödinger dynamics. In fact the light particles are coupled through the heavy one, in the sense that the dynamics is not factorized in any coordinate system.

References

[1] Joos E and Zeh H D 1985 The emergence of classical properties through interaction with the environment *Z. Phys.* B **59** 223–43

[2] Gallis M R and Fleming G N 1990 Environmental and spontaneous localization *Phys. Rev.* A **42** 38–48

[3] Tegmark M 1993 Apparent wave function collapse caused by scattering *Found. Phys. Lett.* **6** 571–90

[4] Hornberger K and Sipe J E 2003 Collisional decoherence reexamined *Phys. Rev.* A **68** 012105

[5] Giulini D, Joos E, Kiefer C, Kupsch J, Stamatescu I-O and Zeh H D 1996 *Decoherence and the Appearance of a Classical World in Quantum Theory* (Berlin: Springer)

[6] Blanchard Ph, Giulini D, Joos E, Kiefer C and Stamatescu I-O (ed) 2000 *Decoherence: Theoretical, Experimental and Conceptual Problems (Lect. Notes in Phys.* vol 538*)* (Berlin: Springer)

[7] Dell'Antonio G F 2003 On decoherence *J. Math. Phys.* **44** 4939–56

[8] Dürr D, Figari R and Teta A 2004 Decoherence in a two-particle model *J. Math. Phys.* **45** 1291–309

[9] Adami R, Figari R, Finco D and Teta A 2004 On the asymptotic behaviour of a quantum two-body system in the small mass ratio limit *J. Phys A: Math. Gen.* **37** 7567–80

[10] Albeverio S, Gesztesy F, Hoegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (Berlin: Springer)

[11] Reed M and Simon B 1979 *Methods of Modern Mathematical Physics: III. Scattering Theory* (New York: Academic)

[12] Simon B 1971 *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms* (Princeton, NJ: Princeton University Press)

[13] Dell'Antonio G and Panati G 2004 The flux-across-surfaces theorem and zero-energy resonances *J. Stat. Phys.* **116** 1161–80

[14] Scarlatti S and Teta A 1990 Derivation of the time-dependent propagator for the three-dimensional schrödinger equation with one point interaction *J. Phys. A: Math. Gen.* **23** L1033–L1035

[15] Albeverio S, Brzeźniak Z and Dabrowski L 1995 Fundamental Solution of the heat and Schrödinger equations with point interaction *J. Funct. Anal.* **130** 220–54

[16] Yajima K 1995 The $W^{k,p}$-continuity of wave operators for Schrödinger operators *J. Math. Soc. Japan* **47** 551–81